

DEFORMED UNIVERSAL CHARACTERS FOR CLASSICAL AND AFFINE ALGEBRAS

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ABSTRACT. Creation operators are given for the three distinguished bases of the type BCD universal character ring of Koike and Terada yielding an elegant way of treating computations for all three types in a unified manner. Deformed versions of these operators create symmetric function bases whose expansion in the universal character basis, has polynomial coefficients in q with non-negative integer coefficients. We conjecture that these polynomials are one-dimensional sums associated with crystal bases of finite-dimensional modules over quantized affine algebras for all nonexceptional affine types. These polynomials satisfy a Macdonald-type duality.

CONTENTS

1. Introduction	2
1.1. Universal characters of classical type	2
1.2. Creation operators	2
1.3. K polynomials	2
1.4. The $X = M = K$ conjecture	3
1.5. Dual bases and affine Kazhdan-Lusztig polynomials	3
1.6. Outline of Paper	4
2. Plethystic formulae	4
3. Root systems and characters	7
4. Bases of symmetric functions	9
5. Creation operators for the bases s_λ^\diamond and determinants	13
5.1. The Schur basis	13
5.2. Creating the bases s^\diamond	15
5.3. Determinantal formulae	15
5.4. Kernels	18
6. Hall-Littlewood symmetric functions and analogues	19
6.1. Deformed Schur basis	19
6.2. Deformed s_λ^\diamond basis	20
7. Parabolic Hall-Littlewood operators and analogues	22
7.1. Definition of operators	22
7.2. \mathbb{H}^\diamond in terms of \mathbb{H}^\varnothing	23
7.3. Connection between \mathbb{B}^\diamond and \mathbb{H}^\diamond	24
8. $X = M = K$	27

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1. INTRODUCTION

1.1. Universal characters of classical type. It is well-known that the ring Λ of symmetric functions is the universal character ring of type A . That is, for every $n \in \mathbb{Z}_{>0}$ there is a ring epimorphism $\Lambda \rightarrow R(GL(n))$ from Λ onto the ring of polynomial representations of $GL(n)$, which sends the Schur function s_λ to the isomorphism class of the irreducible $GL(n)$ -module of highest weight λ .

The ring Λ is also the universal character ring for types B , C , and D . Using identities of Littlewood [12], Koike and Terada constructed two distinguished bases $\{s_\lambda^\square\}$ and $\{s_\lambda^{\square\square}\}$ of Λ corresponding to the irreducible characters of the symplectic and orthogonal groups. These two bases have the same structure constants $d_{\lambda\mu\nu}$.

There is a third basis $\{s_\lambda^\diamond\}$ of Λ that also has structure constants $d_{\lambda\mu\nu}$ which appears in [10]. This basis is implicitly defined in [6] where it was also shown that up to a natural constraint involving Schur function expansions, the only bases of Λ with structure constants $d_{\lambda\mu\nu}$ are the three bases $\{s_\lambda^\diamond\}$ for $\diamond \in \{\square, \square\square, \square\}$.

1.2. Creation operators. Bernstein's creation operator B_r [28] is a linear endomorphism of Λ . The operators B_r create the Schur basis in the sense that $B_{\lambda_1} B_{\lambda_2} \cdots B_{\lambda_k} 1 = s_\lambda$ where $\lambda = (\lambda_1, \dots, \lambda_k)$. The primary purpose of this paper is to study the creation operators B_r^\diamond for the bases $\{s_\lambda^\diamond\}$ and their q -analogues \tilde{B}_r^\diamond . The q -analogues are defined by the general construction of the q -analogue of any symmetric function operator [27].

1.3. K polynomials. Jing [4] showed that the type A operators \tilde{B}_r create the modified Hall-Littlewood symmetric functions, whose Schur function expansion coefficients are the Kostka-Foulkes polynomials. More generally, there is a parabolic analogue \tilde{B}_ν of \tilde{B}_r which yields the generalized Kostka polynomials [25].

We consider the \diamond -analogues of these operators. The operators \tilde{B}_r^\diamond create symmetric functions, which, when expanded in the basis s_λ^\diamond , yield coefficient polynomials $d_{\lambda\mu}(q)$ with nonnegative integer coefficients. These polynomials are the same for $\diamond \in \{\square, \square\square, \square\}$ and are q -analogues of the multiplicity of the irreducible character χ^λ in the tensor product $\bigotimes_i V(\mu_i \omega_1)$ of type C_n irreducibles indexed by multiples of the first fundamental weight.

One may consider the straightforward parabolic analogue B_ν^\diamond of the creation operator B_r^\diamond and its q -analogue \tilde{B}_ν^\diamond . The operators \tilde{B}_ν^\diamond yield polynomials that fail to have nonnegative coefficients.

Motivated by considerations detailed below, for each $\diamond \in \{\emptyset, \square, \blacksquare, \boxplus\}$ we define an \diamond -variant H_ν^\diamond of the type A operator \tilde{B}_ν . These operators create symmetric functions whose expansion into the basis s_λ^\diamond , has coefficients $K^\diamond \in \mathbb{Z}_{\geq 0}[q]$. The polynomials K^\diamond include the polynomials $d_{\lambda\mu}(q)$ as a special case and we show that all K^\diamond are linear combinations of the $d(q)$ coefficients.

The creation operators give an explicit formula for K^\diamond in terms of the generalized Kostka polynomials. The generalized Kostka polynomials satisfy a Macdonald-type duality [19] [21]. This implies a duality for the polynomials K^\diamond which, as a special case, relates the polynomials K^\boxplus and K^\blacksquare .

1.4. The $X = M = K$ conjecture. Let \mathfrak{g} be a Kac-Moody Lie algebra of nonexceptional affine type, with derived subalgebra \mathfrak{g}' and simple Lie subalgebra $\bar{\mathfrak{g}}$. Let $U_q(\mathfrak{g}) \supset U'_q(\mathfrak{g}) \supset U_q(\bar{\mathfrak{g}})$ be the corresponding quantized universal enveloping algebras. The papers [2] [3] assert the existence of a certain family of finite-dimensional $U'_q(\mathfrak{g})$ -modules $W_s^{(k)}$ called Kirillov-Reshetikhin (KR) modules. The KR modules are conjectured to have crystal bases $B^{k,s}$. It is expected that all irreducible finite-dimensional $U'_q(\mathfrak{g})$ -modules that have a crystal base, are tensor products of KR-modules. Such modules have the structure of a graded $U_q(\bar{\mathfrak{g}})$ -module. Their graded multiplicities are called one-dimensional sums X . The above authors also define the fermionic formula M , whose form is suggested by the Bethe Ansatz, and conjecture that $X = M$.

As the rank of \mathfrak{g} goes to infinity, the fermionic formula M stabilizes. We call these polynomials the stable fermionic formulae. Of all the infinite families of affine root systems with distinguished 0 node, there are only four different families of stable fermionic formulae. These four families M^\diamond for $\diamond \in \{\emptyset, \square, \blacksquare, \boxplus\}$ correspond to the four bases given by the Schur functions $s_\lambda^\emptyset = s_\lambda$ and the bases s_λ^\diamond . The family \diamond associated with a given affine root system, is determined by the part of the Dynkin diagram near the 0 node.

We conjecture that $X = M = K$ for the stable formulae (large rank case). In type $A_n^{(1)}$ this follows by combining the papers [9] [20] [25]. It is also known to hold for $q = 1$. In a separate publication [23] it will be shown that $X = K$ holds for types $\diamond \in \{\square, \blacksquare\}$ for tensor factors of the form $B^{1,s}$ using the virtual crystal theory of [17] and a generalization of the Schensted bijection for oscillating tableaux due to Delest, Dulucq, and Favreau [1].

1.5. Dual bases and affine Kazhdan-Lusztig polynomials. The dual bases to $\{s_\lambda^\diamond\}$ for $\diamond \in \{\emptyset, \square, \boxplus, \blacksquare\}$ with respect to the standard scalar product are families which are elements of the completion of the ring of symmetric functions. They are defined as the bases $\{s_\lambda^{\diamond*}\}$ with the property $\langle s_\mu^\diamond, s_\lambda^{\diamond*} \rangle = \delta_{\lambda\mu}$. This implies that $s_\lambda^{\diamond*}$ for $\diamond = \emptyset, \square, \boxplus, \blacksquare$ (respectively) is a Schur function times $\prod_{\alpha \in \bar{\Phi}^+} (1 - e^\alpha)^{-1}$ (with $x_i = e^{\epsilon_i}$) where $\bar{\Phi}^+$ are the set

of positive roots of type A , B , C , D (respectively) which are not positive roots of type A .

In an interesting connection we find that creation operators for the $s_\lambda^{\diamond*}$ bases can also be used to define linearly independent sets of elements related to the Weyl groups of type A_n , B_n , C_n and D_n which have Lusztig's weight multiplicity polynomials as change of basis coefficients. These results will be explored further in a separate paper.

1.6. Outline of Paper. We begin in Section 2 with the derivation of symmetric function identities using the ‘plethystic’ notation à la Garsia. Section 3 introduces notation for classical root systems and translates expressions involving root systems into the language of symmetric functions.

In section 4 we give a plethystic definition for the bases $\{s_\lambda^\diamond\}$ based on Littlewood's formulae. The bases $\{s_\lambda^\diamond\}$ for $\diamond \in \{\emptyset, \blacksquare, \boxplus, \boxminus\}$ are defined as special cases of a single formula involving skewing (dual to multiplication) operators acting on Schur functions allowing a unified account of formulas in the rest of the paper. An important advantage of our definition, is that any homomorphism on symmetric functions which transform a basis of one type to another is easily realized as a substitution on the alphabet of the symmetric function.

Such algebra homomorphisms allow us to give explicit formulae for creation operators B_r^\diamond for the bases $\{s_\lambda^\diamond\}$ in section 5. These creation operators are used to derive several Jacobi-Trudi like identities for the distinguished bases, in particular recovering some identities of Weyl and the determinantal definition of the bases $\{s_\lambda^{\blacksquare}\}$ and $\{s_\lambda^{\boxplus}\}$ given in [11]. We also derive creation operators for the dual bases to s_λ^\diamond .

In section 6 we study the straightforward q -analogues \tilde{B}_ν^\diamond of the parabolic creation operators B_ν^\diamond . Section 7 gives the definition of the K polynomials via the \diamond -analogues H_ν^\diamond of the parabolic Hall-Littlewood creation operator \tilde{B}_ν .

Section 8 states the $X = M = K$ conjecture.

2. PLETHYSTIC FORMULAE

Let Λ be the ring of symmetric functions, to which we apply the ‘plethystic notation.’ We refer the reader to [15] for symmetric function identities and we will list all the necessary additional identities we will use in this section but we will not include a complete exposition on this notation.

Assume that the letters X, Y, Z and W represent sums of monomials with coefficient 1 so that XY represents a product of these sums of monomials. Expressions like $x \in X$ refers to x being a single monomial in the multiset of this expression.

Λ has a scalar product $\langle \cdot, \cdot \rangle$ with respect to which the Schur functions $\{s_\lambda\}$ are an orthonormal basis. The reproducing kernel for this scalar product is $\Omega[XY]$, where

$$\Omega[X - Y] = \frac{\prod_{y \in Y} (1 - y)}{\prod_{x \in X} (1 - x)} = \left(\sum_{r \geq 0} (-1)^r s_{(1^r)}[Y] \right) \left(\sum_{r \geq 0} s_r[X] \right). \quad (2.1)$$

Cauchy's formula expands the kernel in terms of Schur functions:

$$\Omega[XY] = \sum_{\lambda} s_{\lambda}[X] s_{\lambda}[Y]. \quad (2.2)$$

Given $P[X] \in \Lambda$, the skewing operator $P[X]^{\perp} \in \text{End}(\Lambda)$ is the linear operator that is adjoint to multiplication by $P[X]$ with respect to the scalar product. In other words, for all $P[X] \in \Lambda$,

$$P[X]^{\perp}(\Omega[XY]) = \Omega[XY]P[Y]. \quad (2.3)$$

Taking $P[X] = \Omega[XZ]$ and skewing in the X variables, we have

$$\Omega[XZ]^{\perp}(\Omega[XY]) = \Omega[XY]\Omega[YZ] = \Omega[(X + Z)Y]. \quad (2.4)$$

Remark 1. All skewing operations in this paper are done with respect to symmetric functions in the X variables only. Expressions which contain other sets of variables are considered as coefficients of the linear operator. In addition, consider $V, W \in \text{End}(\Lambda)$ that do not involve the arbitrary set of variables Y , if $V(\Omega[XY]) = W(\Omega[XY])$, then by (2.2) and by taking the coefficient of $s_{\lambda}[Y]$ it holds that $V(s_{\lambda}[X]) = W(s_{\lambda}[X])$. Since $s_{\lambda}[X]$ is a basis for Λ , then $V = W$ as elements of $\text{End}(\Lambda)$.

Due to the previous remark we have that for all $P[X] \in \Lambda$,

$$\Omega[XZ]^{\perp}(P[X]) = P[X + Z]. \quad (2.5)$$

We compute the commutation of the multiplication operator $\Omega[XZ]$ and the skewing operator $\Omega[XW]^{\perp}$, letting the composite operators act on $\Omega[XY]$.

$$\begin{aligned} \Omega[XW]^{\perp}(\Omega[XZ]\Omega[XY]) &= \Omega[XW]^{\perp}(\Omega[X(Y + Z)]) \\ &= \Omega[(X + W)(Y + Z)] \\ &= \Omega[(X + W)Z]\Omega[XW]^{\perp}(\Omega[XY]). \end{aligned}$$

This yields the operator identity

$$\Omega[XW]^{\perp} \circ \Omega[XZ] = \Omega[(X + W)Z] \circ \Omega[XW]^{\perp}. \quad (2.6)$$

In particular, for all $P[X] \in \Lambda$,

$$\Omega[XW]^{\perp} \circ P[X] = P[X + W] \circ \Omega[XW]^{\perp}. \quad (2.7)$$

The comultiplication map $\Delta : \Lambda \rightarrow \Lambda \otimes \Lambda$ may be computed as follows. Let $P \in \Lambda$, expand $P[X + Y]$ as a sum of products of the form $P_1[X]P_2[Y]$: $P[X + Y] = \sum_{(P)} P_1[X]P_2[Y]$. Then $\Delta(P) = \sum_{(P)} P_1 \otimes P_2$.

The adjoint operators P^\perp act on products by [15, Ex. I.5.25(d)]

$$P^\perp(QR) = \sum_{(P)} P_1^\perp(Q)P_2^\perp(R). \quad (2.8)$$

Given any operator $V \in \text{End}(\Lambda)$, one of the authors [27] defined its t -analogue $\tilde{V} \in \text{End}(\Lambda)$ by

$$\tilde{V}(P[X]) = V^Y(P[tX + (1-t)Y])|_{Y \rightarrow X} \quad (2.9)$$

where V^Y acts on the Y variables and $Y \rightarrow X$ is the substitution map.

We now apply the above construction for $V = \Omega[XZ] \circ \Omega[XW]^\perp$. We have that for $P[X] \in \Lambda$,

$$\Omega[YZ]\Omega[YW]^\perp P[tX + (1-t)Y] = \Omega[YZ]P[tX + (1-t)(Y+W)].$$

Then

$$\begin{aligned} \tilde{V}(P[X]) &= \Omega[XZ]P[tX + (1-t)(X+W)] \\ &= \Omega[XZ]P[X + (1-t)W] = \Omega[XZ]\Omega[XW(1-t)]^\perp P[X]. \end{aligned}$$

By linearity, for all $P[X], Q[X] \in \Lambda$, if $V = P[X] \circ Q[X]^\perp$, then

$$\tilde{V} = P[X] \circ Q[X(1-t)]^\perp. \quad (2.10)$$

For this V , at $t = 0$ the operator V is recovered:

$$\tilde{V}|_{t=0} = P[X] \circ Q[X]^\perp = V. \quad (2.11)$$

At $t = 1$, the operator

$$\tilde{V}|_{t=1} = P[X]Q[0] \quad (2.12)$$

is multiplication by $P[X]Q[0]$.

For later use we compute the commutation of the operator $\Omega[s_{(12)}[X]]^\perp$ with a multiplication operator.

Proposition 2.

$$\Omega[Ws_{(12)}[X]]^\perp \circ \Omega[ZX] = \Omega[ZX + Ws_{(12)}[Z]]\Omega[W(s_{(12)}[X] + ZX)]^\perp. \quad (2.13)$$

Proof.

$$\begin{aligned} &\Omega[Ws_{(12)}[X]]^\perp (\Omega[ZX]\Omega[XY]) \\ &= \Omega[Ws_{(12)}[X]]^\perp (\Omega[(Y+Z)X]) \\ &= \Omega[(Y+Z)X]\Omega[Ws_{(12)}[Y+Z]] \\ &= \Omega[(Y+Z)X]\Omega[W(s_{(12)}[Y] + YZ + s_{(12)}[Z])] \\ &= \Omega[ZX + Ws_{(12)}[Z]]\Omega[XY]\Omega[W(s_{(12)}[Y] + YZ)] \\ &= \Omega[ZX + Ws_{(12)}[Z]](\Omega[W(s_{(12)}[X] + ZX)]^\perp (\Omega[XY])). \end{aligned}$$

□

3. ROOT SYSTEMS AND CHARACTERS

In this section we fix notation for the root systems of classical type and recall Littlewood's plethystic formula and Weyl's determinantal formulae for the Weyl denominator.

Consider a root system Φ of type A_{n-1} , B_n , C_n , or D_n , coming from the Lie group GL_n , O_{2n+1} , Sp_{2n} , or O_{2n} respectively. We use GL_n instead of SL_n for convenience.

Let $I = \{1, 2, \dots, n-1\}$ for type A_{n-1} and $I = \{1, 2, \dots, n\}$ for types B_n, C_n, D_n . Let ϵ_i be the i -th standard basis vector of \mathbb{Z}^n . A set of simple roots $\Delta = \{\alpha_i \mid i \in I\}$ can be given by

$$\Delta = \begin{cases} \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq n-1\} & \text{for type } A_{n-1} \\ \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\alpha_n = \epsilon_n\} & \text{for type } B_n \\ \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\alpha_n = 2\epsilon_n\} & \text{for type } C_n \\ \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\alpha_n = \epsilon_{n-1} + \epsilon_n\} & \text{for type } D_n \end{cases}$$

The simple roots form a basis for the real Euclidean space E that they span. For type A_{n-1} , $E = \{v \in \mathbb{R}^n \mid \langle v, \epsilon \rangle = 0\}$ where $\epsilon = (1, 1, \dots, 1) \in \mathbb{R}^n$. For types B_n, C_n, D_n , $E = \mathbb{R}^n$.

The simple coroots $\{\alpha_i^\vee \mid i \in I\} \subset E$ are defined by $\alpha_i^\vee = \frac{2}{\langle \alpha_i, \alpha_i \rangle} \alpha_i$ for $i \in I$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product on \mathbb{R}^n .

The fundamental weights $\{\omega_i \mid i \in I\} \subset E^*$ are defined by the condition that they be a dual basis to the coroot basis of E with respect to the standard scalar product on \mathbb{R}^n . The weight lattice is defined by $P = \bigoplus_{i \in I} \mathbb{Z} \omega_i \subset E^*$ and the dominant weights are given by $P^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \omega_i$. In type A_{n-1} , $E^* \cong \mathbb{R}^n / \mathbb{R} \epsilon$. For types B_n, C_n, D_n we shall make the identification $E^* = E$.

The dominant weights are given by

$$\begin{cases} \{\omega_i = (1^i, 0^{n-i}) \bmod \mathbb{R} \epsilon \mid 1 \leq i \leq n-1\} & \text{for type } A_{n-1} \\ \{\omega_i = (1^i, 0^{n-i}) \mid 1 \leq i \leq n-1\} \cup \{\omega_n = (1/2)\epsilon\} & \text{for type } B_n \\ \{\omega_i = (1^i, 0^{n-i}) \mid 1 \leq i \leq n-1\} \cup \{\omega_n = \epsilon\} & \text{for type } C_n \\ \{\omega_i = (1^i, 0^{n-i}) \mid 1 \leq i \leq n-2\} \cup \\ \quad \{\omega_{n-1} = ((1/2)^{n-1}, -1/2), \omega_n = (1/2)\epsilon\} & \text{for type } D_n \end{cases}$$

For $i \in I$ let $s_i \in GL(E)$ be the linear map given by the reflection through the hyperplane orthogonal to α_i : $s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$ for $i \in I$ and $\lambda \in E$. The Weyl group W is the group generated by $\{s_i \mid i \in I\}$. For type A_{n-1} , W is the symmetric group S_n on n symbols. For types B_n and C_n , W is the hyperoctahedral group of signed permutations on n symbols. For type D_n W is the subgroup of the hyperoctahedral group on n symbols consisting of the signed permutations with an even number of negative signs.

The root system Φ is the W -orbit of Δ in E . The set Φ^+ of positive roots is defined by

$$\Phi^+ = \{\alpha \in \Phi \mid \langle \alpha, \omega_i \rangle \geq 0 \text{ for all } i \in I. \}$$

Explicitly

$$\Phi^+ = \begin{cases} \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\} & \text{for type } A_{n-1} \\ \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{\epsilon_i \mid 1 \leq i \leq n\} & \text{for type } B_n \\ \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{2\epsilon_i \mid 1 \leq i \leq n\} & \text{for type } C_n \\ \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n\} & \text{for type } D_n \end{cases}$$

Let $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{i \in I} \omega_i$. Explicitly

$$\rho = \begin{cases} (n-1, n-2, \dots, 1, 0) \pmod{\mathbb{R}\epsilon} & \text{for type } A_{n-1} \\ (1/2)(2n-1, 2n-3, \dots, 3, 1) & \text{for type } B_n \\ (n, n-1, \dots, 2, 1) & \text{for type } C_n \\ (n-1, n-2, \dots, 1, 0) & \text{for type } D_n \end{cases}$$

Let $\mathbb{Z}[P]$ be the group \mathbb{Z} -algebra of the weight lattice, with \mathbb{Z} -basis e^λ for $\lambda \in P$. The Weyl group W acts on P and therefore on $\mathbb{Z}[P]$. Let $J = \sum_{w \in W} \varepsilon(w)w$ be the W -antisymmetrization operator on $\mathbb{Z}[P]$. Define the linear map π on $\mathbb{Z}[P]$ by $\pi f = J(e^\rho)^{-1} J(e^\rho f)$.

The Weyl character formula says that the character χ^λ of the irreducible module of highest weight $\lambda \in P^+$, is given by

$$\chi^\lambda = \pi(e^\lambda). \quad (3.1)$$

We define this also for any $\lambda \in P$. For any $w \in W$ we have

$$\chi^\lambda = \varepsilon(w) \chi^{w(\lambda+\rho)-\rho}. \quad (3.2)$$

It follows that $\chi^\lambda \neq 0$ if and only if $\lambda + \rho$ has trivial stabilizer in W . In this case there is a unique $w \in W$ such that $w(\lambda + \rho) \in P^+$. Thus χ^λ is either zero or an irreducible character, up to sign.

The Weyl denominator (up to the factor $e^{-\rho}$) is defined by

$$\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) = e^{-\rho} J(e^\rho) \quad (3.3)$$

Let $z_i = e^{\epsilon_i}$ for $1 \leq i \leq n$, $Z = (z_1, z_2, \dots, z_n)$, $z_i^* = z_i^{-1}$, and $Z^* = z_1^* + z_2^* + \dots + z_n^*$. Then the Weyl denominator for these four cases is given in our notation as

$$\begin{aligned} R^A(Z) &= R(Z) = \prod_{1 \leq i < j \leq n} (1 - z_j z_i^*) \\ R^B(Z) &= R(Z) \Omega[-f_{\square}[Z^*]] \\ R^C(Z) &= R(Z) \Omega[-f_{\blacksquare}[Z^*]] \\ R^D(Z) &= R(Z) \Omega[-f_{\boxtimes}[Z^*]] \end{aligned} \quad (3.4)$$

where we have defined

$$\begin{aligned} f_{\emptyset}[X] &= 0 \\ f_{\square}[X] &= s_1[X] + s_{(1^2)}[X] \\ f_{\blacksquare}[X] &= s_2[X] \\ f_{\boxplus}[X] &= s_{(1^2)}[X] \end{aligned} \quad (3.5)$$

We remark that due to symmetric function identities, we know that $s_1[Z^*] = Z^*$, $s_{(1^2)}[Z^*] = \sum_{i < j} z_i^* z_j^*$ and $s_2[Z^*] = \sum_{i \leq j} z_i^* z_j^*$. Along with equation (2.1), we may arrive at other expressions for $R^X(Z)$, but for manipulations we have chosen to express them as symmetric function identities.

The following is essentially due to Weyl.

Proposition 3. *Consider the algebra automorphism of $\mathbb{Z}[P]$ induced by $z_i \mapsto z_{n+1-i}^*$. Letting \overline{R} denote the image of the Weyl denominator R under this map, we have the $n \times n$ determinantal formulae*

$$\begin{aligned} R(Z) &= \overline{R}^A(Z) = \det |z_i^{i-j}| \\ R(Z)\Omega[-f_{\square}[Z]] &= \overline{R}^B(Z) = \det |z_i^{i-j} - z_i^{i+j-1}| \\ R(Z)\Omega[-f_{\blacksquare}[Z]] &= \overline{R}^C(Z) = \det |z_i^{i-j} - z_i^{i+j}| \\ R(Z)\Omega[-f_{\boxplus}[Z]] &= \overline{R}^D(Z) = \frac{1}{2} \det |z_i^{i-j} + z_i^{i+j-2}| \end{aligned}$$

Proof. Let $\overline{\rho}$ be the reverse of ρ . The algebra automorphism sends $e^{w\rho-\rho}$ to $e^{\overline{\rho}-w\overline{\rho}}$. For type B_n we have that $\overline{\rho} = \frac{1}{2}(1, 3, \dots, 2n-1)$ and

$$\begin{aligned} R(Z)\Omega[-Z_n - s_{(1^2)}[Z_n]] &= \sum_{w \in S_n} \epsilon(w) \sum_{b \in \{\pm 1\}^n} \left(\prod_{i=1}^n b_i \right) z^{\overline{\rho} - \alpha w \overline{\rho}} \\ &= \sum_{w \in S_n} \epsilon(w) \sum_{b \in \{\pm 1\}^n} \prod_{i=1}^n b_i z_i^{i-1/2-b_i(w_i-1/2)} \\ &= \sum_{w \in S_n} \epsilon(w) \prod_{i=1}^n \left(z_i^{i-1/2-w_i+1/2} - z_i^{i-1/2+w_i-1/2} \right) \\ &= \det \left| z_i^{i-j} - z_i^{i+j-1} \right|_{1 \leq i, j \leq n} \end{aligned}$$

The other types are similar. \square

4. BASES OF SYMMETRIC FUNCTIONS

We consider four bases of the algebra Λ of symmetric functions. The first is the basis $\{s_{\lambda}\}$ of Schur functions.

For $\diamond \in \{\emptyset, \square, \blacksquare, \boxplus\}$, define the symmetric function

$$s_{\lambda}^{\diamond}[X] = \Omega[-f_{\diamond}]^{\perp} s_{\lambda}[X] \quad (4.1)$$

with f_\diamond as in (3.5). All of the families $\{s_\lambda^\diamond\}$ are bases of Λ , due to the inverse formula

$$s_\lambda[X] = \Omega[f_\diamond]^\perp s_\lambda^\diamond[X]. \quad (4.2)$$

Of course $s_\lambda^\emptyset = s_\lambda$ is the basis of Schur functions, which are the universal characters for the special/general linear groups. The bases $\{s_\lambda^\boxplus\}$ and $\{s_\lambda^{\boxminus\boxplus}\}$ appear in [11] as the universal characters for the symplectic and orthogonal groups respectively; see (4.22). The basis $\{s_\lambda^\boxminus\}$ is not mentioned in [11] but appears implicitly in [6].

To explain the notation for the bases, let \mathcal{P}^\diamond be the set of partitions that can be tiled using the shape \diamond without changing the orientation of the tile \diamond . In other words, $\mathcal{P}^\emptyset = \{\emptyset\}$ is the singleton set containing the empty partition, \mathcal{P}^\boxplus is the set of all partitions, $\mathcal{P}^{\boxminus\boxplus}$ is the set of partitions with even rows, and \mathcal{P}^\boxminus is the set of partitions with even columns.

Littlewood's formulae give the Schur function expansions of $\Omega[\pm f_\diamond]$.

$$\Omega[f_\diamond] = \sum_{\lambda \in \mathcal{P}^\diamond} s_\lambda[X] \quad (4.3)$$

$$\Omega[-f_{\boxplus}] = \sum_{\mu=(\alpha_1, \dots, \alpha_p | \alpha_1-1, \dots, \alpha_p-1)} (-1)^{|\mu|/2} s_\mu[X] = \prod_{i < j} 1 - x_i x_j \quad (4.4)$$

$$\Omega[-f_{\boxminus\boxplus}] = \sum_{\mu=(\alpha_1-1, \dots, \alpha_p-1 | \alpha_1, \dots, \alpha_p)} (-1)^{|\mu|/2} s_\mu[X] = \prod_{i \leq j} 1 - x_i x_j \quad (4.5)$$

$$\Omega[-f_{\boxminus}] = \sum_{\mu=\mu^t} (-1)^{(|\mu|+d(\mu))/2} s_\mu[X] = \prod_{i < j} 1 - x_i x_j \prod_i 1 - x_i, \quad (4.6)$$

where $(\alpha|\beta)$ is Frobenius' notation for a partition and $d(\mu)$ is the size of largest diagonal.

We introduce notation for the linear maps that change among the bases $\{s_\lambda^\diamond\}$ for various \diamond . In plethystic formulae let ε represent a variable that has been specialized to the scalar -1 . We will consider ε a special element with the property $\varepsilon^2 = 1$ and

$$\Omega[\varepsilon X - \varepsilon Y] = \frac{\prod_{y \in Y} (1 + y)}{\prod_{x \in X} (1 + x)} \quad (4.7)$$

For $\diamond, \heartsuit \in \{\emptyset, \boxplus, \boxminus, \boxminus\boxplus\}$ define the linear isomorphism $i_\diamond^\heartsuit : \Lambda \rightarrow \Lambda$ by

$$i_\diamond^\heartsuit(s_\lambda^\diamond[X]) = s_\lambda^\heartsuit[X] \quad (4.8)$$

for all λ . It is given by

$$i_\diamond^\heartsuit = \Omega[f_\diamond - f_\heartsuit]^\perp. \quad (4.9)$$

Proposition 4. *For all $P \in \Lambda$,*

$$i_{\square}^{\square} P[X] = P[X - 1] \quad i_{\square}^{\square} P[X] = P[X + 1] \quad (4.10)$$

$$i_{\square}^{\square} P[X] = P[X - 1 - \varepsilon] \quad i_{\square}^{\square} P[X] = P[X + 1 + \varepsilon] \quad (4.11)$$

$$i_{\square}^{\square} P[X] = P[X - \varepsilon] \quad i_{\square}^{\square} P[X] = P[X + \varepsilon] \quad (4.12)$$

Proof. We have

$$\Omega[f_{\square} - f_{\square}] = \Omega[-X] \quad (4.13)$$

$$\Omega[f_{\square} - f_{\square}] = \Omega[-p_2[X]] = \prod_{x \in X} (1 - x^2) = \Omega[-(1 + \varepsilon)X] \quad (4.14)$$

All of the formulae follow from these and (2.5). \square

In particular, since substitution maps are algebra homomorphisms, one has the following result, which was obtained in [11] for the pair \square and \square .

Corollary 5. $i_{\diamond}^{\heartsuit}$ is an algebra isomorphism for $\diamond, \heartsuit \in \{\square, \square, \square\}$.

Define the structure constants ${}^{\diamond}c_{\mu\nu}^{\lambda}$ by

$$s_{\mu}^{\diamond}[X] s_{\nu}^{\diamond}[X] = \sum_{\lambda} {}^{\diamond}c_{\mu\nu}^{\lambda} s_{\lambda}^{\diamond}[X]. \quad (4.15)$$

The coefficient ${}^{\square}c_{\mu\nu}^{\lambda}$ is the ordinary Littlewood-Richardson coefficient $c_{\mu\nu}^{\lambda}$. By Corollary 5, the other three sets of structure constants coincide (this is proved in [11] for \square and \square); call this common structure constant $d_{\lambda\mu\nu}$.

Kleber [6] showed that the three bases $\{s_{\lambda}^{\diamond}\}$ for $\diamond \in \{\square, \square, \square\}$ are the unique bases with this property, assuming a certain kind of expansion in terms of Schur functions.

Theorem 6. [6] *Suppose $\{v_{\lambda}\}$ is a basis of the ring of symmetric functions such that*

$$v_{\mu} v_{\nu} = \sum_{\lambda} d_{\lambda\mu\nu} v_{\lambda} \quad (4.16)$$

for all μ, ν and that

$$s_{\lambda} \in v_{\lambda} + \sum_{\mu < \lambda} \mathbb{Z}_{\geq 0} v_{\mu} \quad (4.17)$$

where $\mu \leq \lambda$ means that $\mu_1 + \dots + \mu_i \leq \lambda_1 + \dots + \lambda_i$ for all i (but μ and λ need not have the same number of cells). Then $\{v_{\lambda}\}$ must be one of the bases $\{s_{\lambda}^{\square}\}$, $\{s_{\lambda}^{\square}\}$, or $\{s_{\lambda}^{\square}\}$.

[illegible]
$$d_{\lambda\mu\nu} = \sum_{\rho,\sigma,\tau \in \mathcal{P}} c_{\rho\tau}^{\mu} c_{\sigma\tau}^{\nu} c_{\rho\sigma}^{\lambda}. \quad (4.19)$$
$$\begin{aligned}
d_{\lambda\mu\nu} &= \left\langle s_{\mu}^{\Theta} s_{\nu}^{\Theta}, s_{\lambda}^{\Theta} \right\rangle = \left\langle \Omega[s_{(1^2)}[X]]^{\perp} \left(s_{\mu}^{\Theta} s_{\nu}^{\Theta} \right), s_{\lambda} \right\rangle \\
&= \sum_{\tau} \left\langle (\Omega[s_{(1^2)}[X]]^{\perp} s_{\tau}[X]^{\perp} s_{\mu}^{\Theta}) (\Omega[s_{(1^2)}[X]]^{\perp} s_{\tau}[X]^{\perp} s_{\nu}^{\Theta}), s_{\lambda} \right\rangle \\
&= \sum_{\tau} \left\langle (s_{\tau}^{\perp} s_{\mu}) (s_{\tau}^{\perp} s_{\nu}), s_{\lambda} \right\rangle = \sum_{\tau} \left\langle s_{\mu/\tau} s_{\nu/\tau}, s_{\lambda} \right\rangle \\
&= \sum_{\rho, \sigma, \tau} c_{\rho\tau}^{\mu} c_{\sigma\tau}^{\nu} c_{\rho\sigma}^{\lambda}.
\end{aligned} \tag{4.20}$$
$$c_{\mu^t \nu^t}^{\lambda^t} = c_{\mu \nu}^{\lambda}. \quad (4.21)$$

Corollary 10. $d_{\lambda^t \mu^t \nu^t} = d_{\lambda \mu \nu}$.

The Schur functions are self dual with respect to the standard scalar product of the symmetric functions, that is we have $\langle s_\lambda[X], s_\mu[X] \rangle = \delta_{\lambda\mu}$. Since by definition,

$$\langle \Omega[-f_\diamond[X]]^\perp f[X], g[X] \rangle = \langle f[X], \Omega[-f_\diamond[X]]g[X] \rangle,$$

the bases $s_\lambda^\diamond[X] = \Omega[-f_\diamond[X]]^\perp s_\lambda[X]$ will be dual to the functions $s_\lambda^{\diamond*}[X] := \Omega[f_\diamond[X]]s_\lambda[X]$ since

$$\begin{aligned} \langle s_\lambda^{\diamond*}[X], s_\mu^\diamond[X] \rangle &= \langle \Omega[f_\diamond[X]]s_\lambda[X], \Omega[-f_\diamond[X]]^\perp s_\mu[X] \rangle \\ &= \langle s_\lambda[X], \Omega[f_\diamond[X]]^\perp \Omega[-f_\diamond[X]]^\perp s_\mu[X] \rangle \\ &= \langle s_\lambda[X], s_\mu[X] \rangle = \delta_{\lambda\mu}. \end{aligned}$$

The elements $s_\lambda^{\diamond*}[X]$ are elements of the completion of the symmetric functions, $\hat{\Lambda}$, and computation of the expansion in the Schur basis can be computed using equation (4.3).

We shall only consider characters χ^λ where $\lambda \in \mathbb{Z}^n \cap P^+$ using the above realizations of P . The connection between the characters χ^λ and the symmetric functions s_λ^\diamond is due to Littlewood. The character $\chi^\lambda : G \rightarrow \mathbb{C}$, defined by the trace of the action of $g \in G$ on the highest weight module V^λ of G , is a polynomial function in the eigenvalues of g . A typical element of the group G has eigenvalues of the form

$$\begin{cases} (z_1, \dots, z_n) & \text{for } G = GL_n \\ (z_1, \dots, z_n, 1, z_1^{-1}, \dots, z_n^{-1}) & \text{for } G = O_{2n+1} \\ (z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1}) & \text{for } G = Sp_{2n} \text{ or } O_{2n}. \end{cases}$$

With $g \in G$ having the above eigenvalues and $Z = z_1 + \dots + z_n$, we have

$$\chi^\lambda(g) = \begin{cases} s_\lambda^\emptyset[Z] & \text{for } G = GL_n \\ s_\lambda^{\square\square}[Z + 1 + Z^*] & \text{for } G = O_{2n+1} \\ s_\lambda^{\square\square}[Z + Z^*] & \text{for } G = Sp_{2n} \\ s_\lambda^{\square\square}[Z + Z^*] & \text{for } G = O_{2n}. \end{cases} \quad (4.22)$$

Because of relations between the bases we also have $s_\lambda^{\square\square}[Z + Z^* + 1] = \chi^\lambda(g)$ for $G = Sp_{2n}$. Proctor's odd symplectic group Sp_{2n+1} [18], which is neither simple nor reductive, has an indecomposable representation with trace χ^λ , which restricted to elements $g \in Sp_{2n+1}$ of determinant 1, satisfies $\chi^\lambda(g) = s_\lambda^{\square\square}[Z + 1 + Z^*]$.

5. CREATION OPERATORS FOR THE BASES s_λ^\diamond AND DETERMINANTS

5.1. The Schur basis. The Schur functions $\{s_\lambda \mid \lambda \in \mathcal{P}\}$ are the unique family of symmetric functions indexed by partitions, which for $\lambda = (r)$ are

given by

$$s_r[X] = \begin{cases} s_r[X] & \text{if } r \in \mathbb{Z}_{\geq 0} \\ 0 & \text{otherwise,} \end{cases} \quad (5.1)$$

and for $\lambda \in \mathcal{P}$ are given by the Jacobi-Trudi determinant

$$s_\lambda[X] = \det |s_{\lambda_i - i + j}[X]|_{1 \leq i, j \leq \ell(\lambda)} \quad (5.2)$$

where $\ell(\lambda)$ is the number of parts of λ . We may also define $s_\nu[X]$ for $\nu \in \mathbb{Z}^n$ using (5.1) and (5.2). It follows that for all w in the symmetric group $W = S_n$,

$$s_\nu[X] = \varepsilon(w) s_{w(\nu + \rho) - \rho}[X] \quad (5.3)$$

where $\rho = (n-1, n-2, \dots, 1, 0) \in \mathbb{Z}^n$.

We now recall the creation operators for the Schur basis. Bernstein's operators $\{B_r \mid r \in \mathbb{Z}\} \subset \text{End}(\Lambda)$ are defined by the generating function

$$B(z) = \sum_{r \in \mathbb{Z}} B_r z^r = \Omega[zX] \Omega[-z^*X]^\perp \quad (5.4)$$

where $z^* = 1/z$. For $\nu \in \mathbb{Z}^n$, define

$$B_\nu = B_{\nu_1} \circ B_{\nu_2} \circ \dots \circ B_{\nu_n} \in \text{End}(\Lambda).$$

We claim that

$$B_\nu 1 = s_\nu[X]. \quad (5.5)$$

The proof is included since the proofs for the other bases s_λ^\diamond are very similar.

This result follows from a formula for the composition of Bernstein operators. Let $Z = z_1 + z_2 + \dots + z_n$ and $Z^* = z_1^* + \dots + z_n^*$. Define

$$B(Z) = \sum_{\nu \in \mathbb{Z}^n} z^\nu B_\nu = B(z_1) \dots B(z_n). \quad (5.6)$$

We have

$$\begin{aligned} B(z)B(w) &= \Omega[zX] \Omega[-z^*X]^\perp \Omega[wX] \Omega[-w^*X]^\perp \\ &= \Omega[zX] \Omega[w(X - z^*)] \Omega[-z^*X]^\perp \Omega[-w^*X]^\perp \\ &= \Omega[-wz^*] \Omega[(z + w)X] \Omega[-(z^* + w^*)X]^\perp. \end{aligned} \quad (5.7)$$

Iterating (5.7) yields

$$B(Z) = R(Z) \Omega[ZX] \circ \Omega[-Z^*X]^\perp. \quad (5.8)$$

Rewriting $R(Z)$ by Proposition 3 for type A_{n-1} , applying $B(Z)$ to $1 \in \Lambda$ and taking the coefficient at z^ν , we have that

$$\begin{aligned} B_\nu 1 &= \Omega[ZX] \det |z_i^{i-j}|_{1 \leq i, j \leq n} \Big|_{z^\nu} \\ &= \det |s_{\nu_i - i + j}[X]|_{1 \leq i, j \leq n} = s_\nu[X]. \end{aligned} \quad (5.9)$$

5.2. Creating the bases s^\diamond . Define the operators $B_\nu^\diamond \in \text{End}(\Lambda)$ for $\nu \in \mathbb{Z}^n$ by

$$B^\diamond(Z) = \sum_{\nu \in \mathbb{Z}^n} z^\nu B_\nu^\diamond = i_\emptyset^\diamond \circ B(Z) \circ i_\emptyset^\diamond. \quad (5.10)$$

For $\nu \in \mathbb{Z}^n$ it follows from (5.5) and (4.8) that

$$B_{\nu_1}^\diamond \cdots B_{\nu_n}^\diamond 1 = B_\nu^\diamond 1 = s_\nu^\diamond[X]. \quad (5.11)$$

The operator $B^\diamond(Z)$ has the following plethystic formula.

Proposition 11. For $\diamond \in \{\square, \boxplus, \boxtimes\}$,

$$B^\diamond(Z) = R(Z)\Omega[-f_\diamond[Z]]\Omega[ZX]\Omega[-(Z + Z^*)X]^\perp. \quad (5.12)$$

Proof. First let $\diamond = \boxplus$. By (5.8) and (2.13) with $W = -1$ (as an alphabet), we have

$$\begin{aligned} B^{\boxplus}(Z) &= i_\emptyset^{\boxplus} B(Z) i_\emptyset^\emptyset \\ &= \Omega[-s_{(1^2)}[X]]^\perp R(Z)\Omega[ZX]\Omega[-Z^*X]^\perp \Omega[s_{(1^2)}[X]]^\perp \\ &= R(Z)\Omega[-s_{(1^2)}[Z]]\Omega[ZX]\Omega[-ZX]^\perp \Omega[-Z^*X]^\perp \\ &= R(Z)\Omega[-f_{\boxplus}[Z]]\Omega[ZX]\Omega[-(Z + Z^*)X]^\perp. \end{aligned}$$

Next let $\diamond = \square$.

$$\begin{aligned} B^\square(Z) &= i_{\boxplus}^\square B^{\boxplus}(Z) i_{\boxplus}^\square \\ &= \Omega[-X]^\perp R(Z)\Omega[-f_{\boxplus}[Z]]\Omega[ZX]\Omega[-(Z + Z^*)X]^\perp \Omega[X]^\perp \\ &= R(Z)\Omega[-s_{(1^2)}[Z]]\Omega[Z(X - 1)]\Omega[-(Z + Z^*)X]^\perp \\ &= R(Z)\Omega[-f_\square[Z]]\Omega[ZX]\Omega[-(Z + Z^*)X]^\perp. \end{aligned}$$

The proof for $\diamond = \boxtimes$ is similar. \square

It follows that

$$B^\square(Z) = \Omega[-Z]B^{\boxplus}(Z) \quad (5.13)$$

$$B^{\boxtimes}(Z) = \Omega[-(1 + \varepsilon)Z]B^{\boxplus}(Z) \quad (5.14)$$

5.3. Determinantal formulae. Recall that the Schur functions satisfy the Jacobi-Trudi identity (5.2). The other three bases satisfy a common determinantal formula due to Weyl for s^{\boxplus} and s^{\boxtimes} . See [11, Thm. 2.3.3].

Proposition 12. For $\diamond \in \{\square, \boxplus, \boxtimes\}$ the basis $\{s_\lambda^\diamond \mid \lambda \in \mathcal{P}\}$ of Λ is characterized by

$$\begin{aligned} s_r^{\boxplus} &= s_r \\ s_r^\square &= s_r - s_{r-1} \\ s_r^{\boxtimes} &= s_r - s_{r-2} \end{aligned} \quad (5.15)$$

for $r \in \mathbb{Z}$ and

$$s_\lambda^\diamond = \frac{1}{2} \det \left| s_{\lambda_i - i + j}^\diamond + s_{\lambda_i - i - j + 2}^\diamond \right|_{1 \leq i, j \leq \ell(\lambda)} \quad (5.16)$$

Proof. Take $n = \ell(\lambda)$. Using the type D_n denominator formula in Proposition 3, Proposition 11 for $\diamond = \mathbb{B}$, applying $B^\mathbb{B}(Z)$ to $1 \in \Lambda$ and taking the coefficient of $\nu \in \mathbb{Z}^n$ we have that

$$\begin{aligned} s_\nu^\mathbb{B} &= B^\mathbb{B}(Z)1 \Big|_{z^\nu} \\ &= \frac{1}{2} \Omega[ZX] \det \left| z_i^{i-j} + z_i^{i+j-2} \right|_{z^\nu} \\ &= \frac{1}{2} \det |s_{\nu_i - i + j} + s_{\nu_i + i + j - 2}|. \end{aligned}$$

This proves the formula for $\diamond = \mathbb{B}$. For $\diamond \in \{\square, \blacksquare\}$ apply the algebra isomorphism $\iota_\diamond^\mathbb{B}$. \square

Example 13.

$$\begin{aligned} s_{(4,3,3)}^{\blacksquare} &= \frac{1}{2} \begin{vmatrix} 2s_4 & s_5 + s_3 & s_6 + s_2 \\ 2s_2 & s_3 + s_1 & s_4 + 1 \\ 2s_1 & s_2 + 1 & s_3 \end{vmatrix} \\ s_{(4,3,3)}^\mathbb{B} &= \frac{1}{2} \begin{vmatrix} 2(s_4 - s_2) & s_5 - s_1 & s_6 - s_4 + s_2 - 1 \\ 2(s_2 - 1) & s_3 & s_4 - s_2 + 1 \\ 2s_1 & s_2 & s_3 - s_1 \end{vmatrix} \\ s_{(4,3,3)}^\square &= \frac{1}{2} \begin{vmatrix} 2(s_4 - s_3) & s_5 - s_4 + s_3 - s_2 & s_6 - s_5 + s_2 - s_1 \\ 2(s_2 - s_1) & s_3 - s_2 + s_1 - 1 & s_4 - s_3 + 1 \\ 2(s_1 - 1) & s_2 - s_1 + 1 & s_3 - s_2 \end{vmatrix} \end{aligned}$$

Amusingly, if the other denominator identities in Proposition 3 are used, we obtain more determinantal formulae for s_λ^\diamond . For the denominator of type C_n we have the following result. For the case $\diamond = \blacksquare$ see [11, Thm. 2.3.3].

Proposition 14.

$$s_\lambda^\diamond = \det \left| \tilde{s}_{\lambda - i + j}^\diamond - \tilde{s}_{\lambda_i - i - j}^\diamond \right|_{1 \leq i, j \leq \ell(\lambda)}$$

for $\diamond \in \{\square, \mathbb{B}, \blacksquare\}$ where $\tilde{s}_r^\blacksquare = s_r$, $\tilde{s}_r^\mathbb{B} = s_r + s_{r-2} + s_{r-4} + \cdots$, and $\tilde{s}_r^\square = s_r - s_{r-1} + s_{r-2} - \cdots + (-1)^r$.

Example 15.

$$\begin{aligned} s_{(4,3,3)}^{\blacksquare} &= \det \begin{vmatrix} s_4 - s_2 & s_5 - s_1 & s_6 - 1 \\ s_2 - 1 & s_3 & s_4 \\ s_1 & s_2 & s_3 \end{vmatrix} \\ s_{(4,3,3)}^\mathbb{B} &= \det \begin{vmatrix} s_4 & s_5 + s_3 & s_6 + s_4 + s_2 \\ s_2 & s_3 + s_1 & s_4 + s_2 + 1 \\ s_1 & s_2 + 1 & s_3 + s_1 \end{vmatrix} \end{aligned}$$

$$s_{(4,3,3)}^{\square} = \det \begin{vmatrix} s_4 - s_3 & s_5 - s_4 + s_3 - s_2 & s_6 - s_5 + s_4 - s_3 + s_2 - s_1 \\ s_2 - s_1 & s_3 - s_2 + s_1 - 1 & s_4 - s_3 + s_2 - s_1 + 1 \\ s_1 - 1 & s_2 - s_1 + 1 & s_3 - s_2 + s_1 - 1 \end{vmatrix}$$

For the type B_n denominator we have new identities.

Proposition 16.

$$s_{\lambda}^{\diamond} = \det \left| \bar{s}_{\lambda_i - i + j}^{\diamond} - \bar{s}_{\lambda_i - i - j + 1}^{\diamond} \right|_{1 \leq i, j \leq \ell(\lambda)}$$

for $\diamond \in \{\square, \boxplus, \boxminus\}$ where $\bar{s}_r^{\square} = s_r$, $\bar{s}_r^{\boxplus} = s_r + s_{r-1}$, and $\bar{s}_r^{\boxminus} = s_r + s_{r-1} + s_{r-2} + \dots + 1$.

Example 17.

$$\begin{aligned} s_{(4,3,3)}^{\square} &= \det \begin{vmatrix} s_4 - s_3 & s_5 - s_2 & s_6 - s_1 \\ s_2 - s_1 & s_3 - 1 & s_4 \\ s_1 - 1 & s_2 & s_3 \end{vmatrix} \\ s_{(4,3,3)}^{\boxplus} &= \det \begin{vmatrix} s_4 & s_5 + s_4 + s_3 & s_6 + s_5 + s_4 + s_3 + s_2 \\ s_2 & s_3 + s_2 + s_1 & s_4 + s_3 + s_2 + s_1 + 1 \\ s_1 & s_2 + s_1 + 1 & s_3 + s_2 + s_1 + 1 \end{vmatrix} \\ s_{(4,3,3)}^{\boxminus} &= \det \begin{vmatrix} s_4 - s_2 & s_5 + s_4 - s_2 - s_1 & s_6 + s_5 - s_1 - 1 \\ s_2 - 1 & s_3 + s_2 - 1 & s_4 + s_3 \\ s_1 & s_2 + s_1 & s_3 + s_2 \end{vmatrix} \end{aligned}$$

Finally, we note that all these formulae have analogues involving the symmetric functions $s_{(1^k)}$ instead of the functions s_k . Recall the algebra isomorphism defined by the involutive map $\omega : \Lambda \rightarrow \Lambda$ given by $\omega(s_{\lambda}) = s_{\lambda^t}$ where λ^t is the transpose of the partition λ . Define the map $\omega_{\diamond} : \Lambda \rightarrow \Lambda$ by $i_{\varnothing}^{\diamond} \circ \omega \circ i_{\varnothing}^{\varnothing}$. By definition

$$\omega_{\diamond}(s_{\lambda}^{\diamond}) = s_{\lambda^t}^{\diamond}. \quad (5.17)$$

We now rewrite (4.22). Using the definition of s_{λ}^{\diamond} in the variables W , multiplying by $s_{\lambda}[X]$, summing over λ , and (2.3) we have

$$\Omega[-f_{\diamond}[X]]\Omega[WX] = \sum_{\lambda \in \mathcal{P}} s_{\lambda}^{\diamond}[W]s_{\lambda}[X].$$

Multiplying by $\Omega[f_{\diamond}[X]]$, we have

$$\Omega[WX] = \Omega[f_{\diamond}[X]] \sum_{\lambda \in \mathcal{P}} s_{\lambda}^{\diamond}[W]s_{\lambda}[X]. \quad (5.18)$$

Applying this formula with $W = Z, Z + Z^*, Z + 1 + Z^*$ and $\diamond \in \{\varnothing, \boxplus, \boxminus\}$ as in (4.22), setting X to be a set of n variables and restricting the sum to $\lambda \in \mathcal{P}^n$, one obtains Littlewood's Cauchy-type formulae [11, Lemma 1.5.1].

Proposition 18. *The map ω_{\diamond} is an algebra isomorphism satisfying*

$$\omega_{\diamond}(s_{\lambda}^{\diamond}) = \omega(s_{\lambda^t}^{\diamond}). \quad (5.19)$$

where $\varnothing^t = \varnothing$, $\boxplus^t = \boxminus$, $\boxminus^t = \boxplus$, and $\square^t = \square$.

Proof. ω_\diamond is an algebra isomorphism by (5.17) and Corollary 10. To prove (5.19) one may reduce to the case $\lambda = (r)$, since the s_r^\diamond are algebra generators of Λ by Proposition 12. For $\lambda = (r)$, equation (5.19) is easy to check directly using the definitions and the explicit formulae (4.4), (4.5), and (4.6) for $\Omega[-f_\diamond[X]]$. \square

By applying the algebra isomorphism (5.19) to the formulae in Proposition 12 and using (5.19) for $\lambda = (r)$, one obtains determinantal formulae for $s_{\lambda_t}^\diamond$ in terms of $s_{1^r}^{\diamond^t} = \omega(s_r^\diamond)$. These are simple expressions in the $s_{(1^r)}$ symmetric functions, due to (5.15):

$$s_{(1^r)}^\diamond = \begin{cases} s_{(1^r)} & \text{for } \diamond = \emptyset, \square \\ s_{(1^r)} - s_{(1^{r-2})} & \text{for } \diamond = \boxminus \\ s_{(1^r)} - s_{(1^{r-1})} & \text{for } \diamond = \square. \end{cases} \quad (5.20)$$

5.4. Kernels. The Cauchy element $\Omega = \sum_{n \geq 0} s_n$ naturally plays a role in the generating function for the structure coefficients of the ring of symmetric functions. Since $s_\lambda[X + Y] = \sum_{\mu, \nu} c_{\mu\nu}^\lambda s_\mu[X] s_\nu[Y]$ and

$$\Omega[XY] = \sum_{\lambda} s_\lambda[X] s_\lambda[Y], \quad (5.21)$$

then putting these two identities together we have a generating function for all of the LR coefficients:

$$\Omega[X(Y + Z)] = \sum_{\lambda, \mu, \nu} c_{\mu\nu}^\lambda s_\lambda[X] s_\mu[Y] s_\nu[Z]$$

Similarly we have $s_\lambda[XY] = \sum_{\mu, \nu} k_{\lambda\mu\nu} s_\mu[X] s_\nu[Y]$ where $k_{\lambda\mu\nu}$ are the Kronecker coefficients arising the expression $s_\lambda * s_\mu = \sum_{\nu} k_{\lambda\mu\nu} s_\nu$ and $\frac{p_\mu}{z_\mu} * \frac{p_\nu}{z_\nu} = \delta_{\lambda\mu} \frac{p_\lambda}{z_\lambda}$. Substituting YZ for Y in in equation (5.21) yields

$$\Omega[XYZ] = \sum_{\lambda, \mu, \nu} k_{\lambda\mu\nu} s_\lambda[X] s_\mu[Y] s_\nu[Z].$$

The structure coefficients $d_{\lambda\mu\nu}$ satisfy the following property.

Proposition 19.

$$\Omega[XY + XZ + YZ] = \sum_{\lambda, \mu, \nu} d_{\lambda\mu\nu} s_\lambda[X] s_\mu[Y] s_\nu[Z] \quad (5.22)$$

Proof.

$$\begin{aligned} \Omega[XY + XZ + YZ] &= \sum_{\alpha, \beta, \tau} s_\alpha[X] s_\alpha[Y] s_\beta[X] s_\beta[Z] s_\tau[Y] s_\tau[Z] \\ &= \sum_{\alpha, \beta, \tau, \lambda, \mu, \nu} c_{\alpha\beta}^\lambda c_{\alpha\tau}^\mu c_{\beta\tau}^\nu s_\lambda[X] s_\mu[Y] s_\nu[Z] \\ &= \sum_{\lambda, \mu, \nu} d_{\lambda\mu\nu} s_\lambda[X] s_\mu[Y] s_\nu[Z] \end{aligned}$$

□

Both the left and right hand side of the above expression are independent of $\diamond \in \{\boxplus, \boxtimes, \boxdot\}$ and yet the coefficients $d_{\lambda\mu\nu}$ appear in this expression. It turns out that this is only because the kernel $\Omega[XY + XZ + YZ]$ is common to all three bases as we show in the following corollary.

Lemma 20.

$$f_\diamond[X + Y] = f_\diamond[X] + XY + f_\diamond[Y]$$

Proof. Notice that $s_{(1^2)}[X + Y] = s_{(1^2)}[X] + XY + s_{(1^2)}[Y]$, $s_2[X + Y] = s_2[X] + XY + s_2[Y]$ and $s_1[X + Y] + s_{(1^2)}[X + Y] = s_1[X] + s_1[Y] + s_{(1^2)}[X] + XY + s_{(1^2)}[Y]$. Therefore we have for all $\diamond \in \{\boxplus, \boxtimes, \boxdot\}$. □

Corollary 21. For $\diamond \in \{\boxplus, \boxtimes, \boxdot\}$,

$$\Omega[f_\diamond[X + Y + Z]] = \sum_{\lambda, \mu, \nu} d_{\lambda\mu\nu} s_\lambda^{\diamond*}[X] s_\mu^{\diamond*}[X] s_\nu^{\diamond*}[X]$$

Proof. From the previous lemma we can derive that $\Omega[f_\diamond[X + Y]] = \Omega[XY]\Omega[f_\diamond[X]]\Omega[f_\diamond[Y]] = \sum_\lambda s_\lambda^{\diamond*}[X] s_\lambda^{\diamond*}[Y]$. In addition we have,

$$\begin{aligned} \Omega[f_\diamond[X + Y + Z]] &= \Omega[f_\diamond[X] + f_\diamond[Y] + f_\diamond[Z] + XY + XZ + YZ] \\ &= \Omega[f_\diamond[X] + f_\diamond[Y] + f_\diamond[Z]] \sum_{\lambda, \mu, \nu} d_{\lambda\mu\nu} s_\lambda[X] s_\mu[Y] s_\nu[Z] \\ &= \sum_{\lambda, \mu, \nu} d_{\lambda\mu\nu} s_\lambda^{\diamond*}[X] s_\mu^{\diamond*}[Y] s_\nu^{\diamond*}[Z]. \end{aligned}$$

□

6. HALL-LITTLEWOOD SYMMETRIC FUNCTIONS AND ANALOGUES

6.1. Deformed Schur basis. Define

$$\tilde{B}(Z) = \sum_{\nu \in \mathbb{Z}^n} z^\nu \tilde{B}_\nu \quad (6.1)$$

where \tilde{B}_ν is the t -analogue of B_ν from equation (2.9). This is the “parabolic modified” analogue of Jing’s Hall-Littlewood creation operator. It was studied in [25], where it is denoted by H_ν^t . By (5.8) and (2.10),

$$\tilde{B}(Z) = R(Z)\Omega[ZX]\Omega[(t-1)Z^*X]^\perp. \quad (6.2)$$

Let $Z^{(1)}, \dots, Z^{(L)}$ be a family of finite ordered alphabets and R_1 through R_L partitions such that the number of parts of R_j is equal to the number of letters in $Z^{(j)}$ for all j . Define the symmetric functions $\mathbb{B}_R[X; t]$ and polynomials $c_{\lambda; R}(t)$ by

$$\tilde{B}_{R_1} \cdots \tilde{B}_{R_L} 1 = \mathbb{B}_R[X; t] = \sum_\lambda s_\lambda[X] c_{\lambda; R}(t). \quad (6.3)$$

The $c_{\lambda; R}(t)$ are the generalized Kostka polynomials of [24], as proved in [25].

By (2.11) and (5.5) we have

$$\mathbb{B}_R[X; 0] = B_{R_1} \cdots B_{R_L} 1 = s_{(R_1, \dots, R_L)}[X] \quad (6.4)$$

where (R_1, \dots, R_L) denotes the sequence of integers obtained by juxtaposing the parts of the partitions R_j . By (2.12) and (5.5) we have

$$\mathbb{B}_R[X; 1] = s_{R_1}[X] \cdots s_{R_L}[X]. \quad (6.5)$$

Example 22. For $R = (\square, \boxplus, \square)$ we have

$$\begin{aligned} \mathbb{B}_R[X; t] = & \begin{array}{|c|} \hline \square \\ \hline \end{array} + t \begin{array}{|c|} \hline \square \\ \hline \end{array} + t \begin{array}{|c|} \hline \square \\ \hline \end{array} + (t^2 + t) \begin{array}{|c|} \hline \square \\ \hline \end{array} + t^2 \begin{array}{|c|} \hline \square \\ \hline \end{array} \\ & + (t^3 + t^2) \begin{array}{|c|} \hline \square \\ \hline \end{array} + t^3 \begin{array}{|c|} \hline \square \\ \hline \end{array} + t^4 \begin{array}{|c|} \hline \square \\ \hline \end{array} \end{aligned} \quad (6.6)$$

6.2. Deformed s_λ^\diamond basis. Let \tilde{B}_ν^\diamond be the t -analogue of B_ν^\diamond . For $\diamond \in \{\square, \boxplus, \square\}$ define

$$\tilde{B}^\diamond(Z) = \sum_{\nu \in \mathbb{Z}^n} z^\nu \tilde{B}_\nu^\diamond. \quad (6.7)$$

By (2.10), Proposition 11, (5.13) and (5.14),

$$\begin{aligned} \tilde{B}^\boxplus(Z) &= R(Z) \Omega[-s_{(1^2)}[Z]] \Omega[ZX] \Omega[(Z + Z^*)(t-1)X]^\perp \\ \tilde{B}^\square(Z) &= \tilde{B}^\boxplus(Z) \Omega[-Z] \\ \tilde{B}^{\square\square}(Z) &= \tilde{B}^\boxplus(Z) \Omega[-(1+\varepsilon)Z]. \end{aligned} \quad (6.8)$$

For a sequence of partitions $R = (R_1, R_2, \dots, R_L)$, define the symmetric function $\mathbb{B}_R^\diamond[X; t]$ and the polynomials $d_{\lambda R}^\diamond(t)$ by

$$\mathbb{B}_R^\diamond[X; t] = \tilde{B}_{R_1}^\diamond \tilde{B}_{R_2}^\diamond \cdots \tilde{B}_{R_L}^\diamond 1 = \sum_{\lambda} d_{\lambda R}^\diamond(t) s_\lambda^\diamond. \quad (6.9)$$

By (2.11), (2.12), (6.4), (6.5), and (5.10) we have

$$\mathbb{B}_R^\diamond[X; 0] = s_{(R_1, R_2, \dots, R_L)}^\diamond[X] \quad (6.10)$$

$$\mathbb{B}_R^\diamond[X; 1] = s_{R_1}^\diamond[X] s_{R_2}^\diamond[X] \cdots s_{R_L}^\diamond[X]. \quad (6.11)$$

Theorem 23. $d_{\lambda R}^\diamond(t)$ is constant over $\diamond \in \{\square, \boxplus, \square\}$.

Proof. This follows from the fact that the t -analogue operation commutes with the change of basis operations between the kinds $\{\square, \boxplus, \square\}$, when acting on $\tilde{B}^\boxplus(Z)$. For by Propositions 4 and 11 we have

$$\Omega[-X]^\perp \tilde{B}^\boxplus(Z) \Omega[X]^\perp = \Omega[-Z] \tilde{B}^\boxplus(Z) = \tilde{B}^\square(Z) \quad (6.12)$$

$$\Omega[-p_2[X]]^\perp \tilde{B}^\boxplus(Z) \Omega[p_2[X]]^\perp = \Omega[-(1+\varepsilon)Z] \tilde{B}^\boxplus(Z) = \tilde{B}^{\square\square}(Z). \quad (6.13)$$

□

Let us call these polynomials $d_{\lambda R}(t)$. An important special case is when R consists of single-rowed rectangles of sizes given by the partition μ ; in this case write $d_{\lambda \mu}(t)$ instead of $d_{\lambda R}(t)$.

Theorem 24. $d_{\lambda\mu}(t) \in \mathbb{Z}_{\geq 0}[t]$.

These polynomials appear again in Proposition 31 and it is a consequence of this proposition and equation (7.6) that provides a proof of this theorem.

Example 25. We present an example of the symmetric functions created by these operators. Let $\mu = (3, 2, 1)$. For $\diamond \in \{\square, \boxplus, \boxtimes\}$, we will represent the function s_λ^\diamond by the diagram for the partition λ superscripted by \diamond . Notice that our example is independent of \diamond .

$$\begin{aligned} \mathbb{B}_\mu^\diamond[X; t] = & \square^\diamond + t\square^\diamond + t\square^\diamond + (t^2 + t)\square^\diamond + (t^2 + t^3)\square^\diamond \\ & + t^4\square^\diamond + (t^2 + t)\square^\diamond + t\square^\diamond + (2t^2 + t + t^3)\square^\diamond \\ & + (t^4 + t^2 + t^3)\square^\diamond + (t^2 + t^3)\square^\diamond + (t^4 + t^2 + t^3)\square^\diamond + t^4 \end{aligned}$$

One might expect that $d_{\lambda R}(t) \in \mathbb{Z}_{\geq 0}[t]$ if the sequence (R_1, R_2, \dots, R_L) is a partition, since this conjecture has been made for $\diamond = \emptyset$ [25]. An example where this fails to be the case is $R_1 = (3)$, $R_2 = (2, 2)$ and $R_3 = (1)$ and $\diamond = \boxtimes$. There the coefficient of $s_{(1,1)}^\boxtimes$ is $t^5 + t^3 - t^4$.

We now give a more explicit formula for $d_{\lambda\mu}(t)$. Iterating in a manner similar to (5.7), we have

$$\hat{B}^\boxplus(z_1) \cdots \hat{B}^\boxplus(z_n) = \hat{B}^\boxplus(z_1, \dots, z_n) \prod_{1 \leq i < j \leq n} \Omega[tz_j(z_i + z_i^*)] \quad (6.14)$$

Let us apply both sides to the element $1 \in \Lambda$ and take the coefficient at Z^μ . Let n be the length of the partition μ and let Π be the above product over $i < j$. We have

$$\begin{aligned} \mathbb{B}_\mu^\boxplus[X; t] &= R(Z) \Omega[-s_{(1^2)}[Z]] \Omega[ZX] \Pi \Big|_{z^\mu} \\ &= R(Z) \Omega[-s_{(1^2)}[X]]^\perp \Omega[ZX] \Pi \Big|_{z^\mu} \\ &= R(Z) \Pi \sum_{\lambda \in \mathcal{P}^n} s_\lambda[Z] s_\lambda^\boxplus[X] \Big|_{z^\mu}. \end{aligned}$$

Taking the coefficient of $s_\lambda^\boxplus[X]$ for $\lambda \in \mathcal{P}^n$ and using ρ and J of type A_{n-1} we have

$$\begin{aligned} d_{\lambda\mu}(t) &= z^{-\rho} J(z^\rho) s_\lambda[Z] \Pi \Big|_{z^\mu} \\ &= J(z^{\lambda+\rho}) \Pi \Big|_{z^{\mu+\rho}} \\ &= \left(\sum_{w \in S_n} \varepsilon(w) z^{w(\lambda+\rho)} \Pi \right) \Big|_{z^{\mu+\rho}} \\ &= \sum_{w \in S_n} \left(\varepsilon(w) \Pi \Big|_{z^{\mu+\rho-w(\lambda+\rho)}} \right). \end{aligned}$$

Replacing each letter in Z by its reciprocal, we have

$$d_{\lambda\mu}(t) = \sum_{w \in S_n} \varepsilon(w) \prod_{1 \leq i < j \leq n} \Omega[tz_j^*(z_i + z_i^*)] \Big|_{z^{w(\lambda+\rho) - (\mu+\rho)}} \quad (6.15)$$

The product of geometric series can be expanded to obtain a formula resembling Lusztig's t -analogue of weight multiplicity. We also note that

$$\begin{aligned} d_{\lambda\mu}(t) &= \sum_{w \in S_n} \varepsilon(w) z^{\mu+\rho} \prod_{1 \leq i < j \leq n} \Omega[tz_j^*(z_i + z_i^*)] \Big|_{z^{w(\lambda+\rho)}} \\ &= J(z^{\mu+\rho}) \prod_{1 \leq i < j \leq n} \Omega[tz_j^*(z_i + z_i^*)] \Big|_{z^{\lambda+\rho}} \\ &= J^{-1}(z^\rho) J \left(z^{\mu+\rho} \prod_{1 \leq i < j \leq n} \Omega[tz_j^*(z_i + z_i^*)] \right) \Big|_{s_\lambda[Z]}. \end{aligned}$$

7. PARABOLIC HALL-LITTLEWOOD OPERATORS AND ANALOGUES

For each $\diamond \in \{\emptyset, \square, \boxplus, \boxtimes\}$ we define a variant of the same parabolic Hall-Littlewood creation operator. These will be the creation operators for the universal affine characters.

7.1. Definition of operators. Write $\tilde{B}_{t^2}^\diamond(Z)$ for $\tilde{B}^\diamond(Z)$ with t replaced by t^2 . Let

$$H^\diamond(Z) = \sum_{\nu \in \mathbb{Z}^k} z^\nu H_\nu^\diamond = \Omega[f_\diamond[tX] - f_\diamond[X]]^\perp \tilde{B}_{t^2}^\diamond(Z) \Omega[f_\diamond[X] - f_\diamond[tX]]^\perp. \quad (7.1)$$

Proposition 26. For $\diamond \in \{\emptyset, \square, \boxplus, \boxtimes\}$,

$$H^\diamond(Z) = \Omega[f_\diamond[tZ]] \tilde{B}_{t^2}^\diamond(Z). \quad (7.2)$$

Proof. There is nothing to prove for $\diamond = \emptyset$. Consider $\diamond = \boxplus$. By Proposition 2 with $W = t^2 - 1$,

$$\begin{aligned} &\Omega[(t^2 - 1)s_{(1^2)}[X]]^\perp \Omega[ZX] \Omega[(1 - t^2)s_{(1^2)}[X]]^\perp \\ &= \Omega[ZX] \Omega[(t^2 - 1)s_{(1^2)}[Z]] \Omega[(t^2 - 1)ZX]^\perp. \end{aligned}$$

Using (6.2) and (6.8) we have

$$\begin{aligned} H^{\boxplus}(Z) &= \Omega[(t^2 - 1)s_{(1^2)}[X]]^\perp R(Z) \Omega[ZX] \Omega[(t^2 - 1)Z^*X]^\perp \Omega[(1 - t^2)s_{(1^2)}[X]]^\perp \\ &= R(Z) \Omega[ZX] \Omega[(t^2 - 1)s_{(1^2)}[Z]] \Omega[(t^2 - 1)(Z + Z^*)X]^\perp \\ &= \Omega[t^2 s_{(1^2)}[Z]] \tilde{B}_{t^2}^{\boxplus}(Z). \end{aligned}$$

Next let $\diamond = \blacksquare$. We have

$$\begin{aligned}
H^{\blacksquare}(Z) &= \Omega[(t^2 - 1)p_2[X]]^\perp H^{\blacksquare}(Z) \Omega[(1 - t^2)p_2[X]]^\perp \\
&= \Omega[(t - 1)(1 + \varepsilon)X]^\perp \Omega[(t^2 - 1)s_{(1^2)}[Z]] R(Z) \Omega[ZX] \\
&\quad \Omega[(t^2 - 1)(Z + Z^*)X]^\perp \Omega[(1 - t)(1 + \varepsilon)X]^\perp \\
&= \Omega[(t^2 - 1)s_{(1^2)}[Z]] R(Z) \\
&\quad \Omega[Z(X + (t - 1)(1 + \varepsilon))] \Omega[(t^2 - 1)(Z + Z^*)X]^\perp \\
&= \Omega[t^2 s_2[Z]] \Omega[-p_2[Z]] \tilde{B}_{t^2}^{\blacksquare}(Z) \\
&= \Omega[t^2 s_2[Z]] \tilde{B}_{t^2}^{\blacksquare}(Z).
\end{aligned}$$

Finally, for $\diamond = \square$ we have

$$\begin{aligned}
H^{\square}(Z) &= \Omega[(t - 1)X]^\perp H^{\square}(Z) \Omega[(1 - t)X]^\perp \\
&= R(Z) \Omega[t^2 s_{(1^2)}[Z]] \Omega[Z(X + (t - 1))] \Omega[(t^2 - 1)(Z + Z^*)X]^\perp \\
&= \Omega[(t - 1)Z] \Omega[t^2 s_{(1^2)}[Z]] \tilde{B}_{t^2}^{\square}(Z) \\
&= \Omega[tZ + t^2 s_{(1^2)}[Z]] \tilde{B}_{t^2}^{\square}(Z).
\end{aligned}$$

□

Let $R = (R_1, R_2, \dots, R_L)$ be a sequence of partitions. For $\diamond \in \{\emptyset, \square, \blacksquare, \blacksquare\}$ define $\mathbb{H}_R^\diamond[X; t]$ and $K_{\lambda; R}^\diamond(t)$ by

$$\mathbb{H}_R^\diamond[X; t] = \sum_{\lambda} K_{\lambda; R}^\diamond(t) s_{\lambda}^\diamond[X] = H_{R_1}^\diamond H_{R_2}^\diamond \cdots H_{R_L}^\diamond 1. \quad (7.3)$$

From the corresponding properties of the operator $\tilde{B}(Z)$, one obtains the specializations at $t = 0$ and $t = 1$, for all \diamond .

$$\mathbb{H}_R^\diamond[X; 0] = s_{(R_1, R_2, \dots, R_L)}^\diamond[X] \quad (7.4)$$

$$\mathbb{H}_R^\diamond[X; 1] = s_{R_1}[X] s_{R_2}[X] \cdots s_{R_L}[X]. \quad (7.5)$$

Remark 27. For any \diamond , by (7.5) $\mathbb{H}_R^\diamond[X; t]$ is a t -deformation of the product of Schur functions. Note also that $K_{\lambda; R}^\diamond(t) = c_{\lambda; R}(t^2)$ where the latter is a generalized Kostka polynomial; see (6.3).

7.2. \mathbb{H}^\diamond in terms of \mathbb{H}^\emptyset . Let $|R| = \sum_i |R_i|$. Observe that

$$\mathbb{H}_R^\diamond[X; t] = \Omega[f_\diamond[tX] - f_\diamond[X]]^\perp \mathbb{H}_R^\emptyset[X; t].$$

From this and the fact that $K_{\tau; R}^\emptyset(t) = 0$ unless $|R| = |\tau|$, it follows that for $\diamond \in \{\emptyset, \square, \blacksquare, \blacksquare\}$,

$$K_{\lambda; R}^\diamond(t) = t^{|R| - |\lambda|} \sum_{\substack{\tau \in \mathcal{P} \\ |\tau| = |R|}} K_{\tau; R}^\emptyset(t) \sum_{\substack{\mu \in \mathcal{P}^\diamond \\ |\mu| = |R| - |\lambda|}} c_{\lambda\mu}^\tau. \quad (7.6)$$

Remark 28. Let $R = (R_1)$ be a single rectangle. For $\mu \subset R_1$ let $\tilde{\mu}$ be the partition obtained by the 180 degree rotation of the skew shape R_1/μ . Then $c_{\lambda\mu}^{R_1} = 0$ unless $\lambda = \tilde{\mu}$, in which case the coefficient is 1. We have

$$\mathbb{H}_{(R_1)}^\diamond[X; t] = \sum_{\substack{\mu \in \mathcal{P}^\diamond \\ \mu \subseteq R_1}} t^{|\mu|} s_\mu^\diamond[X]. \quad (7.7)$$

The known transpose symmetry of the generalized Kostka polynomials for sequences of rectangles, induces a symmetry for all the polynomials $K_{\lambda; R}^\diamond(t)$. Let $||R|| = \sum_{i < j} |R_i \cap R_j|$, $\emptyset^t = \emptyset$, $\square^t = \square$, $\boxplus^t = \boxplus$, and $\boxminus^t = \boxminus$.

Proposition 29. Let R be a dominant sequence of rectangles (that is, one whose widths weakly decrease) and R' a dominant rearrangement of R^t . Then for all partitions λ ,

$$K_{\lambda^t; R'}^{\diamond^t}(t) = t^{2(||R|| + |R| - |\lambda|)} K_{\lambda; R}^\diamond(t^{-1}). \quad (7.8)$$

Proof. For $\diamond = \emptyset$ the formula holds by combining [25], which connects the creation operator formula with the graded character definition of [24], [20], which connects the definition of [24] with a tableau formula, and either [21] or [19], which show that the tableau formula satisfies the above transpose symmetry. In the other cases the formula holds by the case $\diamond = \emptyset$, (7.6), and (4.21). \square

7.3. Connection between \mathbb{B}^\diamond and \mathbb{H}^\diamond . We now make explicit the connection between the polynomials $d_{\lambda R}(t)$ and $K_{\lambda; R}^\diamond(t)$ that is implied by Proposition 26.

Theorem 30. For $\diamond \in \{\emptyset, \square, \boxplus, \boxminus\}$, and $R = (R_1, R_2, \dots, R_k)$ a sequence of partitions,

$$\mathbb{H}_R^\diamond[X; t] = \sum_{\nu^{(i)} \in \mathcal{P}^\diamond} \sum_{\gamma^{(i)}} t^{|\nu|} \left(\prod_i c_{\nu^{(i)} \gamma^{(i)} - (a_i^{b_i})}^{R_i - (a_i^{b_i})} \right) \mathbb{B}_\gamma^\diamond[X; t^2] \quad (7.9)$$

where the sum is over all sequences of partitions $\nu = (\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(k)})$, $b_i = \ell(R_i)$, and $\gamma = (\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(k)})$ is a sequence of $\gamma^{(i)} \in \mathbb{Z}^{b_i}$ with $\gamma^{(i)}$ weakly decreasing and $a_i = \gamma_{b_i}^{(i)}$.

Proof. Recall that we have the relation $\tilde{B}_\nu^\diamond = \epsilon(w) \tilde{B}_{w(\nu+\rho)-\rho}^\diamond$ for $w \in S_n$ and for every $\nu \in \mathbb{Z}^n$ there is a $w \in S_n$ such that $w(\nu + \rho) - \rho = \alpha \in \mathbb{Z}^n$ with $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$. Therefore

$$\tilde{B}^\diamond(Z) = \sum_\nu z^\nu \tilde{B}_\nu^\diamond = \sum_\alpha \sum_{w \in S_n} \epsilon(w) z^{w(\alpha+\rho)-\rho} \tilde{B}_\alpha^\diamond \quad (7.10)$$

where the sum on the right is over all $\alpha \in \mathbb{Z}^n$ with α a weakly decreasing sequence. From equation $R(Z) = z^{-\rho} J(z^\rho)$ and hence

$$R(Z)^{-1} \tilde{B}^\diamond(Z) = \sum_\alpha J(z^\rho)^{-1} J(z^{\alpha+\rho}) \tilde{B}_\alpha^\diamond$$

Now if $\alpha_n = r$ then $z^{w(\alpha+\rho)-\rho} = z^{(r^n)} z^{w(\alpha-(r^n)+\rho)-\rho}$ where (r^n) is shorthand notation for the sequence with r repeated n times. Therefore

$$R(Z)^{-1} \tilde{B}^\diamond(Z) = \sum_{\alpha} z^{(\alpha_n^n)} s_{\alpha-(\alpha_n^n)}[Z] \tilde{B}_{\alpha}^\diamond.$$

We know by proposition 26 that $H^\diamond(Z) = \Omega[f_\diamond[tZ]] \tilde{B}_t^\diamond(Z)$ and since $\Omega[f_\diamond[tZ]] = \sum_{\lambda \in \mathcal{P}^\diamond} t^{|\lambda|} s_\lambda[Z]$ we have that

$$R(Z)^{-1} H^\diamond(Z) = \sum_{\alpha} \sum_{\nu \in \mathcal{P}^\diamond} \sum_{\gamma} z^{(\alpha_n^n)} t^{|\nu|} c_{\alpha-(\alpha_n^n), \nu}^\gamma s_\gamma[Z] \tilde{B}_{\alpha}^{t^2 \diamond}.$$

The sum over γ in this expression will be over all $\gamma \in \mathcal{P}$ with the length of γ less than or equal to n . Multiplying both sides of this equation by $R(Z)$ and using again the identity that $R(Z) s_\lambda[Z] = \sum_{w \in S_n} \epsilon(w) z^{w(\lambda+\rho)-\rho}$ yields

$$H^\diamond(Z) = \sum_{\alpha} \sum_{\nu \in \mathcal{P}^\diamond} \sum_{\gamma} \sum_{w \in S_n} t^{|\nu|} c_{\alpha-(\alpha_n^n), \nu}^\gamma \epsilon(w) z^{w(\gamma+\rho+(\alpha_n^n))-\rho} \tilde{B}_{\alpha}^{t^2 \diamond}.$$

Since the sum is over $\gamma \in \mathcal{P}$, only for w equal to the identity will $w(\gamma + \rho + (\alpha_n^n)) - \rho$ be a partition and the coefficient of z^R for $R \in \mathcal{P}$ will be

$$H_R^\diamond = \sum_{\alpha} \sum_{\nu \in \mathcal{P}^\diamond} t^{|\nu|} c_{\alpha-(\alpha_n^n), \nu}^{R-(\alpha_n^n)} \tilde{B}_{\alpha}^{t^2 \diamond}$$

The theorem follows since for a sequence of rectangles $R = (R^{(1)}, R^{(2)}, \dots, R^{(k)})$ with $R^{(i)} \in \mathcal{P}$ we have $\mathbb{H}_R^\diamond[X; t] = H_{R^{(1)}}^\diamond H_{R^{(2)}}^\diamond \cdots H_{R^{(k)}}^\diamond 1$ and iterating the above formula yields (7.9). \square

Proposition 31. *Let μ be a partition and R the sequence of single-rowed partitions (μ_i) . Then*

$$\mathbb{H}_R^\square[X; t] = \mathbb{B}_R^\square[X; t^2] \quad (7.11)$$

or equivalently,

$$K_{\lambda; R}^\square(t) = d_{\lambda R}(t^2). \quad (7.12)$$

Proof. This follows from Proposition 26 and the fact that $s_{(1^2)}[z] = 0$ for z a single letter. \square

Example 32. We choose $R = (\boxplus, \boxminus)$ as a sequence of rectangles that will index an example of the 4 types of symmetric functions that we define here. The first example $H_R^\emptyset[X; t]$ is a generating function for generalized Kostka polynomials; see Remark 27.

$$\mathbb{H}_{(\boxplus, \boxminus)}^\emptyset[X; t] = \boxplus + t^2 \boxminus$$

$$\mathbb{H}_{(\boxplus, \boxminus)}^\square[X; t] = \boxplus^\square + t^2 \boxminus^\square + t^2 \boxplus^\square + (t^2 + t^4) \boxplus^\square + (t^4 + t^6) \boxminus^\square$$

$$\mathbb{H}_{(\boxplus, \square)}^{\square}[X; t] = \boxplus^{\square} + t^2 \boxplus \boxplus^{\square} + (t^2 + t^4) \boxplus^{\square} + t^4 \boxplus \boxplus^{\square} + (t^4 + t^6) \boxplus \boxplus^{\square}$$

$$\begin{aligned} \mathbb{H}_{(\boxplus, \square)}^{\square}[X; t] &= \boxplus^{\square} + t^2 \boxplus \boxplus^{\square} + t \boxplus^{\square} + (t + t^3) \boxplus^{\square} + t^3 \boxplus \boxplus^{\square} \\ &\quad + t^2 \boxplus^{\square} + 2(t^2 + t^4) \boxplus^{\square} + t^4 \boxplus \boxplus^{\square} + (t^3 + 2t^5) \boxplus \boxplus^{\square} \\ &\quad + (2t^3 + t^5) \boxplus^{\square} + 2(t^4 + t^6) \boxplus^{\square} + (t^5 + t^7) \end{aligned}$$

Next we list the functions $\mathbb{B}_{\gamma}^{\diamond}[X; t^2]$ that are relevant to the computation of $\mathbb{H}_R^{\diamond}[X; t]$ for $\diamond \in \{\boxplus, \boxtimes, \square\}$. For $\diamond = \emptyset$ we have that $\mathbb{B}_R^{\emptyset}[X; t^2] = \mathbb{H}_R^{\emptyset}[X; t]$.

$$\mathbb{B}_{((2,2),(1))}^{\diamond}[X; t^2] = t^2 s_{3,2}^{\diamond} + s_{2,2,1}^{\diamond} + t^2 s_{2,1}^{\diamond}$$

$$\mathbb{B}_{((2,1),(1))}^{\diamond}[X; t^2] = t^2 s_2^{\diamond} + t^2 s_{1,1}^{\diamond} + t^2 s_{3,1}^{\diamond} + t^2 s_{2,2}^{\diamond} + s_{2,1,1}^{\diamond}$$

$$\mathbb{B}_{((2,0),(1))}^{\diamond}[X; t^2] = t^2 s_3^{\diamond} + t^2 s_{2,1}^{\diamond} + t^2 s_1^{\diamond}$$

$$\mathbb{B}_{((1,1),(1))}^{\diamond}[X; t^2] = t^2 s_{2,1}^{\diamond} + s_{1,1,1}^{\diamond} + t^2 s_1^{\diamond}$$

$$\mathbb{B}_{((1,0),(1))}^{\diamond}[X; t^2] = t^2 + t^2 s_2^{\diamond} + t^2 s_{1,1}^{\diamond}$$

$$\mathbb{B}_{((0,0),(1))}^{\diamond}[X; t^2] = t^2 s_1^{\diamond}$$

$$\mathbb{B}_{((2,-1),(1))}^{\diamond}[X; t^2] = (t^2 - 1) s_2^{\diamond}$$

$$\mathbb{B}_{((1,-1),(1))}^{\diamond}[X; t^2] = (t^2 - 1) s_1$$

$$\mathbb{B}_{((0,-1),(1))}^{\diamond}[X; t^2] = (t^2 - 1)$$

$$\mathbb{B}_{(\lambda, (0))}^{\diamond}[X; t^2] = s_{\lambda}^{\diamond}$$

Observe that Theorem 30 says we will have the following relationship.

$$\mathbb{H}_{(\boxplus, \square)}^{\square}[X; t] = \mathbb{B}_{((2,2),(1))}^{\square} + t^2 \mathbb{B}_{((1,1),(1))}^{\square} + t^4 \mathbb{B}_{((0,0),(1))}^{\square}$$

$$\mathbb{H}_{(\boxtimes, \square)}^{\square}[X; t] = \mathbb{B}_{((2,2),(1))}^{\square} + t^2 \mathbb{B}_{((2,0),(1))}^{\square} + t^4 \mathbb{B}_{((0,0),(1))}^{\square}$$

$$\begin{aligned}
\mathbb{H}_{(\boxplus, \boxminus)}^\square[X; t] = & \mathbb{B}_{((2,2),(1))}^\square + t\mathbb{B}_{((2,1),(1))}^\square + t^2\mathbb{B}_{((1,1),(1))}^\square + t^2\mathbb{B}_{((2,0),(1))}^\square \\
& + t^3\mathbb{B}_{((1,0),(1))}^\square + t^4\mathbb{B}_{((0,0),(1))}^\square + t\mathbb{B}_{((2,2),(0))}^\square + t^2\mathbb{B}_{((2,1),(0))}^\square \\
& + t^3\mathbb{B}_{((1,1),(0))}^\square + t^4\mathbb{B}_{((2,0),(0))}^\square + t^4\mathbb{B}_{((1,0),(0))}^\square + t^5\mathbb{B}_{((0,0),(0))}^\square \\
& + t^3\mathbb{B}_{((2,-1),(1))}^\square + t^4\mathbb{B}_{((1,-1),(1))}^\square + t^5\mathbb{B}_{((0,-1),(1))}^\square
\end{aligned}$$

8. $X = M = K$

Consider a nonexceptional affine Lie algebra \mathfrak{g} of type $X_N^{(r)}$, with canonical simple Lie subalgebra $\bar{\mathfrak{g}}$ of rank n . Let λ be a partition whose number of parts is sufficiently smaller than n . There is a standard way to view λ as a dominant integral weight of $\bar{\mathfrak{g}}$, namely, that λ is identified with the weight $\sum_{i=1}^{n-1}(\lambda_i - \lambda_{i+1})\omega_i$ where ω_i is the i -th fundamental weight of $\bar{\mathfrak{g}}$ (with n large enough so that no spin weights occur). Let R be a sequence of rectangles $R = (R_1, R_2, \dots)$ where R_i has r_i rows and s_i columns where r_i is sufficiently smaller than n . Let B^{r_i, s_i} be the crystal graph of the KR module $W_s^{(r)}$ [2] [3]. Let $\overline{X}_{R, \lambda}(t)$ be the classically restricted one-dimensional sum for the crystal

$$B^R = \bigotimes_i B^{r_i, s_i} \quad (8.1)$$

at the isotypic component of the irreducible $U_q(\bar{\mathfrak{g}})$ -crystal of highest weight λ , using the normalized coenergy function. Let $\overline{M}_{R, \lambda}(t)$ be the corresponding fermionic formula, denoted by $M(R, \lambda, t^{-1})$ in [2] [3]. In those two papers it is conjectured that $\overline{X} = \overline{M}$. By analyzing the fermionic formulae one can show the following.

Lemma 33. *There is a well-defined limiting polynomial*

$$\lim_{n \rightarrow \infty} \overline{M}_{R, \lambda}(t) \quad (8.2)$$

as the rank n goes to infinity. It depends only on R , λ , and the affine family of $X_N^{(r)}$. Moreover, there are only four distinct families of such polynomials, which shall be named as follows.

- (1) For $A_n^{(1)}: \overline{M}_{R, \lambda}^\emptyset(t)$.
- (2) For $B_n^{(1)}$, $D_n^{(1)}$, and $A_{2n-1}^{(2)}: \overline{M}_{R, \lambda}^\square(t)$.
- (3) For $C_n^{(1)}$ and $A_{2n}^{(2)\dagger}: \overline{M}_{R, \lambda}^{\square\square}(t)$.
- (4) For $D_{n+1}^{(2)}$ and $A_{2n}^{(2)}: \overline{M}_{R, \lambda}^{\square\square\square}(t)$.

Note that the families are grouped according to the decomposition of $B^{r, s}$ as a $U_q(\bar{\mathfrak{g}})$ -crystal, or equivalently, according to the attachment of the 0 node. See the appendices of [3] [2].

Conjecture 34. *For R a dominant sequence of rectangles and for all $\diamond \in \{\emptyset, \square, \boxplus, \boxtimes\}$,*

$$K_{\lambda;R}^{\diamond}(t) = \overline{M}_{R^t,\lambda^t}^{\diamond^t}(t^{2/\epsilon}) \quad (8.3)$$

where $\epsilon = 1$ except for $\diamond = \square$ in which case $\epsilon = 2$.

This conjecture is surprising: it proposes a simple relationship (coming from the deformations of the type A Hall-Littlewood creation operators) between the type A universal affine characters, and those of all other nonexceptional affine types. At $t = 1$ this was essentially known [6]. However the formulae for the powers of t occurring in the affine characters given either by the fermionic formula or the one-dimensional sum, do not at all suggest such a simple relationship. Perhaps the virtual crystal methods of [17] can be used to prove Conjecture 34.

Equation (8.3) holds for $\diamond = \emptyset$. The proof connects five formulae for the generalized Kostka polynomials.

- (1) The modified Hall-Littlewood creation operator formula, or equivalently, the K polynomials [25].
- (2) Their definition (call it χ_q) as isotypic components of graded Euler characteristic characters of modules supported in the nullcone [24].
- (3) A tableau formula (call it LR) using Littlewood-Richardson tableaux and a generalized charge statistic [19] [20].
- (4) The affine crystal theoretic formula X given by the one-dimensional sum of type $A_n^{(1)}$ [19] [22].
- (5) The fermionic formula M [7].

We have $K = \chi_q$ [25], $\chi_q = LR$ [20], $LR = X$ [19] [22], and $X = M$ [9].

Equation (8.3) also holds for a single rectangle in all nonexceptional affine types, due to the agreement of [2, Appendix A] and [3, Appendix A] with the formula in Remark 28.

Observe that by combining Conjecture 34 and Proposition 29 one obtains the following conjecture.

Conjecture 35.

$$\overline{M}_{R;\lambda}^{\diamond}(t) = t^{\epsilon(|R|+|R|-|\lambda|)} \overline{M}_{R^t,\lambda^t}^{\diamond^t}(t^{-1}). \quad (8.4)$$

This was proved in [8] via a direct bijection for $\diamond = \emptyset$. This is striking as it relates the fermionic formulae of different types. This kind of relation is not at all apparent from the structure of the fermionic formulae.

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